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AN IMPLICIT FORM OF UPSTREAM DIFFERENCING AND ITS APPLICATION T--ETC(U)  
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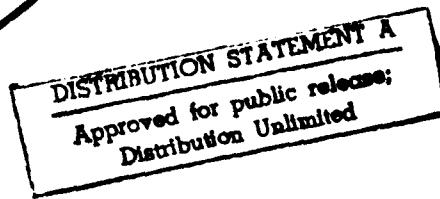
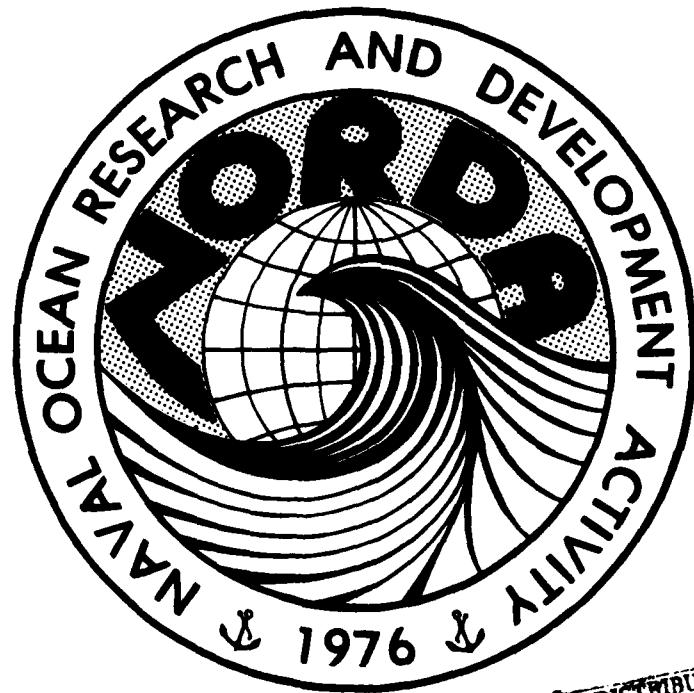
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Naval Ocean Research  
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NSTL Station, Mississippi 39529

# An Implicit Form of Upstream Differencing and Its Application to a Radiation Boundary Condition

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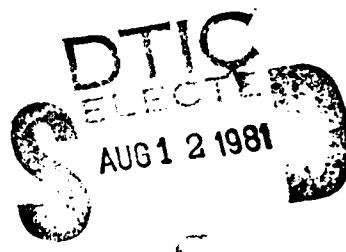
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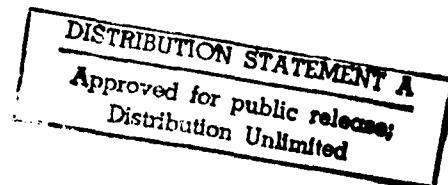
An Implicit Form of Upstream Differencing  
and Its Application to a Radiation Boundary Condition

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July 1981



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## ABSTRACT

An implicit form of upstream differencing is developed which is decoupled in space and is unconditionally stable. It has the same numerical diffusion as the usual explicit form. The scheme is especially attractive for use in radiation boundary conditions for semi-implicit models.

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This note examines an alternate form of upstream differencing which is implicit. Although it could be used as a scheme for advection in a finite difference model, our interest is motivated by a need for an implicit radiation boundary condition. We wish to use a Sommerfeld radiation condition of the form

$$u_t + (u + c) u_x = 0 \quad (1)$$

where  $u$  is a velocity component,  $c$  is a gravity wave speed, and the subscripts are time and space derivatives (see Klemp and Lilly, 1978). In an explicit model this can be implemented by employing the usual explicit form of upstream differencing. In a model where gravity waves are treated implicitly it is likely that the time step would violate the criterion for linear stability of an explicit upstream scheme. An implicit upstream scheme which was unconditionally stable would be of great value in this situation. However, the model design requires further restrictions on the formulation of such a scheme, if it is to be useful in the model. Namely, it must be possible to calculate the boundary values implicitly prior to and independent of the calculation of the interior values at the new time level (see Hurlburt and Thompson, 1980). That is, the implicit scheme must be decoupled in space. We will attempt to design an implicit upstream differencing formula for Equation (1) which satisfies all the preceding requirements.

To examine upstream differencing consider the advection-diffusion equation

$$u_t + uu_x - Au_{xx} = 0 \quad (2)$$

which we difference in space using centered schemes which are second-order accurate:

$$u_t \approx -\frac{u_1}{2\Delta x} (u_{j+1} - u_{j-1}) + \frac{A}{\Delta x^2} (u_{j+1} - 2u_j + u_{j-1}) \quad (3)$$

where  $j$  is a grid index and  $\Delta x$  is a grid increment for  $u$ . If we let the coefficient of the diffusion term equal the coefficient of the advective term, then

$$A = \frac{1}{2} u_j \Delta x \quad (4)$$

and Equation (3) becomes

$$u_t \approx -\frac{u_j}{\Delta x} (u_j - u_{j-1}) \quad (5)$$

Equation (5) is the usual explicit form of upstream differencing for advection, if a forward (Euler) time difference is used. It is first-order accurate in space and time. Equations (3) and (4) show that numerical diffusion is implicit in Equation (5). Table 1 shows the equivalent eddy viscosity for different values of  $u$  and  $\Delta x$  using (4).

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Table 1: Numerical eddy viscosity for upstream differencing

A	u	$\Delta x$
$10^8 \text{ cm}^2/\text{s}$	100 cm/s	20 km
$10^7 \text{ cm}^2/\text{s}$	10 cm/s	20 km
$10^6 \text{ cm}^2/\text{s}$	1 cm/s	20 km

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In mesoscale ocean eddy problems with  $\Delta x = 20$  km and  $u_{\max} \sim 100$  cm/s, typically  $A = 3 \times 10^6 \text{ cm}^2/\text{s}$  has been used. Hence, upstream differencing is highly diffusive. It is less diffusive than  $A = 3 \times 10^6 \text{ cm}^2/\text{s}$  when  $\Delta x = 20$  km only where  $|u| < 3$  cm/s.

For a leapfrog time difference, the advective term in Equation (3) is conditionally stable; but the diffusive term is unstable, and the upstream difference is unstable. For an Euler time difference, the advective term is unstable and the diffusion term is conditionally stable. The upstream difference is conditionally stable ( $\Delta t \leq \Delta x/u$ ) for  $u > 0$ . Note that forming the upstream difference from Equation (3) requires that  $u_{j+1}$  and  $u_{j-1}$  be at the same time level in both the advective and diffusive terms. However, this is not required for  $u_j$ . Nor does it require  $u_j$  at the same time level as  $u_{j+1}$  and  $u_{j-1}$ . Hence, it is possible to use leapfrog for the advective term and Dufort Frankel for the diffusive term. In this case (3) becomes

$$u_j^{n+1} = u_j^{n-1} - \frac{2\Delta t u_j^n}{2\Delta x} (u_{j+1}^n - u_{j-1}^n) + \frac{2\Delta t A}{\Delta x^2} (u_{j+1}^n - u_j^{n+1} - u_j^{n-1} + u_{j-1}^n) \quad (6)$$

where  $n$  is a time step index and  $\Delta t$  is a time increment. If, as before, we use Equation (4), then

$$u_j^{n+1} = u_j^{n-1} - \frac{2u_j^n \Delta t}{\Delta x} \left[ \frac{1}{2} (u_j^{n+1} + u_j^{n-1}) - u_{j-1}^n \right] \quad (7)$$

Equation (7) is an implicit upstream differencing formula which is decoupled in space as desired for the radiation boundary condition in our semi-implicit ocean model. Although second order accurate in time, Equation (7) is first order in space and has the same numerical diffusion as Equation (5). Also, Equation (7) is a three time level scheme whereas Equation (5) required only two time levels. Thus, Equation (7) has a real practical advantage over Equation (5) only if it is unconditionally stable. Since in Equation (6) the advective term is conditionally stable and the diffusive term is unconditionally stable, we anticipate that Equation (7) is at least conditionally stable. Since Equation (7) is implicit and since the

stability of the diffusive term in Equation (3) was dominant when  $u > 0$ , we have some hope for unconditional stability for Equation (7). We will test this hypothesis using a linear stability analysis.

Consider the linerized advection equation

$$u_t = -Uu_x \quad (8)$$

represented by the difference equation, Equation (7). We linearize Equation (7) by letting  $U=u_j^n$  in the coefficient of the space difference. Also let

$$u_j^n = u_n e^{ikj\Delta x} \quad (9)$$

and substitute in Equation (7) to get

$$u_{n+1} = u_{n-1} - \frac{U\Delta t}{\Delta x} (u_{n+1} + u_{n-1} - 2u_n e^{-ik\Delta x}) \quad (10)$$

Let

$$B = U\Delta t / \Delta x \quad (11)$$

the Courant number, and let

$$\alpha = k\Delta x \quad (12)$$

Then Equation (10) can be written

$$u_{n+1} = \frac{2B}{1+B} u_n e^{-i\alpha} + \left( \frac{1-B}{1+B} \right) u_{n-1} \quad (13)$$

We need to write Equation (13) so that the coefficients form an amplification matrix,  $G$ , i.e.,

$$u_{n+1} = Gu_n \quad (14)$$

Thus, Equation (13) becomes

$$\begin{bmatrix} u_{n+1} \\ u_n \\ u_{n-1} \end{bmatrix} = \begin{bmatrix} \frac{2B}{1+B} e^{-i\alpha} & \frac{1-B}{1+B} & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_n \\ u_{n-1} \end{bmatrix} \quad (15)$$

If the spectral radius,  $\sigma$  of  $G \leq 1$ , then  $u_n$  is bounded as  $n \rightarrow \infty$ , where  $\sigma = \max(\lambda_m)$  and the  $\lambda_m$  are the eigenvalues of the matrix. We calculate the eigenvalues using the characteristic equation obtained from  $\det(\lambda I - G) = 0$ , where  $I$  is the identity matrix. The characteristic equation for Equation (15) is

$$\lambda^2 - \frac{2B}{1+B} e^{-i\alpha} \lambda - \frac{1-B}{1+B} = 0 \quad (16)$$

which has roots

$$\lambda = \frac{B}{1+B} e^{-i\alpha} \pm \left[ \left( \frac{B}{1+B} e^{-i\alpha} \right)^2 + \frac{1-B}{1+B} \right]^{\frac{1}{2}} \quad (17)$$

The value of  $|\lambda|$  is maximum when  $e^{-i\alpha} = +1$ . The use of both is redundant, so we consider only  $e^{-i\alpha} = 1$ . For the positive root

$$\max |\lambda| = 1 \quad (18)$$

for the negative

$$\max |\lambda| = \left| \frac{B-1}{B+1} \right| \quad (19)$$

From Equations (11), (18), and (19), Equation (7) is unconditionally stable for  $U \geq 0$  (upstream differencing) and is unstable for  $U < 0$  (downstream differencing).

Thus, an implicit upstream difference has been formulated which is unconditionally stable and is decoupled in space. It meets all the requirements for implementation in a radiation boundary condition in our semi-implicit ocean model.

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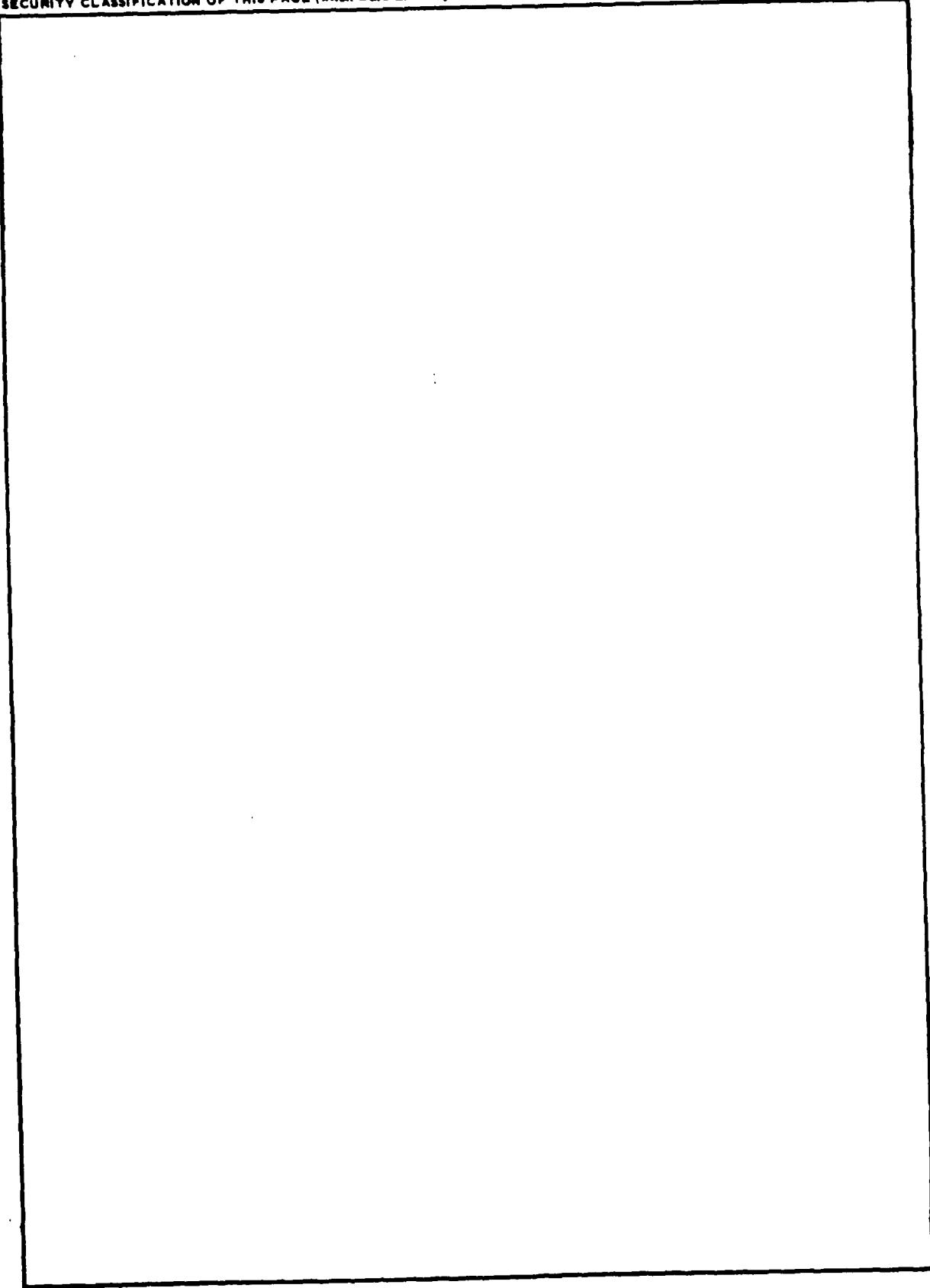
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